

Using Proportional Reasoning Appropriately: Lessons Learned Through Probability

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The middle school years are critical in the development of students' proportional reasoning. Proportional reasoning goes beyond solving the typical "missing value" problem and is essential in a variety of numeric, algebraic, and geometric problems. The National Council of Teachers of Mathematics (NCTM, 2000) suggests that "students' facility with rational numbers and proportionality can be developed in concert with their study of many topics in the middle-grades curriculum" (p. 214).

The study of probability is rich in the use of rational numbers and proportions; however, the nature of probability as a study of random phenomenon makes the use of proportional reasoning somewhat different than in many numeric, algebraic, and geometric contexts. Consider the marble task in figure 1 and the hospital task in figure 2. Before reading on, take a moment to identify the proportional situation embedded in each task and predict how students might respond to each task.

Students' Proportional Reasoning in Probability Contexts

Although each task contains a proportional situation, students do not always use appropriate reasoning based on probabilistic reasoning. Many students answer the tasks in figures 1 and 2 incorrectly, albeit for different reasons. Students often identify Bag B in figure 1 as having the greater chance of obtaining a black marble because there are more black marbles in the bag. Using an intuitive rule that Stavy and Tirosh call "More A-More B" (2000), students often generalize that more black marbles indicate more

chance for black to be chosen. Although the ratio of white to black marbles is equivalent in these bags, students do not usually reason proportionally to determine that the chance of picking a black marble is the same in each bag. On the other hand, students often recognize the proportional situation in the Hospital Task in figure 2 and tasks such as the following:

When flipping a fair coin, which is more likely to occur or are they equally likely?

- a) 2 heads and 1 tail with 3 tosses
- b) 200 heads and 100 tails with 300 tosses

In both the Hospital Task and this coin toss task, students incorrectly reason that the results are equally likely to occur since the distributions are proportional (e.g., $2/3=200/300$). Stavy and Tirosh attribute this erroneous error to an intuitive rule based on “Same A- Same B” where the students assume that if the results are in the same proportion, then the chance of each result occurring is the same. Students’ use of this intuitive rule in this type of problem seems to become more prominent as students get older and their ability to recognize proportionality stabilizes.

Students’ ability to understand and recognize equivalence, either informally (with words or pictures) or formally (with symbols), is going to affect their ability to reason appropriately about probabilities. As Stavey and Tirosh point out, research has shown adolescents tend to not recognize the proportional relationship in the task in figure 1 but yet overgeneralize the proportionality in tasks like the coin toss question above or the Hospital Task in figure 2.

Aside from the intuitive rules theory presented by Stavey and Tirosh, there may be additional reasons that lead students to such erroneous responses. At the upper elementary and middle school level, an underlying concept of equivalence and multiplicative reasoning is important for generating an equivalent class of fractions or ratios, determining whether two fractions or ratios are equivalent, and determining whether two events are equally likely to occur. Many school experiences with equivalent ratios are presented and solved in symbolic form, often devoid of contextual meaning. Without having students analyze situations for equivalent ratios in concrete forms (e.g., the marble bags in figure 1) and contextual situations, students may develop a procedural understanding for determining equivalence of fractions and ratios without truly developing appropriate proportional reasoning that is robust across contexts.

The use of rational numbers and proportional reasoning in probability is often treated as strictly a computation to determine the probability of an event occurring based on a classical approach to probability or to numerically compare theoretical probabilities. Such an approach to probability usually reduces the study of chance to determining and comparing probabilities based on known distributions in a situation (e.g., Bag A in figure has a $\frac{1}{4}$ chance for selecting a white marble) and rarely gives students an opportunity to compare theoretical probabilities with empirical probabilities based on experimental data. Having students compare theoretical probabilities in a numerical manner can lead to an overgeneralization of this technique when students compare the empirical data in the Hospital Task in figure 1.

Building Proportional Reasoning Through Probability Simulations

The use of simulations in the teaching of probability has certainly been advocated and implemented for many years in school curricula. However, without the use of flexible computer tools, such simulations can be time-consuming and result in little depth in the analysis of data. The NCTM (2000) supports the use of simulations in the Statistics and Data Analysis Standard:

Simulations afford students access to relatively large samples that can be generated quickly and modified easily. Technology can thus facilitate students' learning of probability in at least two ways: With large samples, the sample distribution is more likely to be "close" to the actual population distribution, thus reducing the likelihood of incorrect inferences based on empirical samples. With easily generated samples, students can focus on the analysis of the data rather than be distracted by the demands of data collection (p. 253).

Recent research and development (author, 2000) has led to the design of a flexible simulation tool for students in grades 4-9. A purposeful attempt was made to design tools in the *Probability Explorer (PE)*, author, 1999-2002) software that would help students connect concepts in probability with rational numbers and proportional reasoning. Given that students have such difficulty with questions such as the Marble Task and Hospital Task, I have several suggested simulation tasks that can be used to help students understand probability concepts in a richer and more meaningful way and can encourage the development of appropriate proportional reasoning.

Sample Task 1: Compare Probability with Three Bags

Many typical probability tasks ask students to compare the likelihood of an event occurring from a theoretical perspective (e.g., Marble Task in figure 1). The focus on theoretical probability reduces any reasoning involved in such a task to counting and comparing ratios or fractions; thus losing the aspect of the task that is probabilistic in nature. Instead of having students calculate and compare probabilities theoretically, simulation tools such as *PE* allow students to approach such a task empirically. This empirical approach maintains the probabilistic nature of the situation and can provide a novel way to have students compare probabilities and use proportional reasoning.

Consider posing the following task for small groups of students to investigate:

There are three bags of marbles: Bag X contains 1 white and 3 black marbles, Bag Y contains 2 white and 3 black marbles, and Bag Z contains 2 white and 6 black marbles. Make a prediction of which bag you think has the best chance for picking a black marble and discuss a reason for your prediction. Then use *PE* to model each of the bags of marbles and run several simulations of picking a marble from the bag with replacement. Based on the data you collect, make an argument for which bag has the best chance of picking a black marble.

Many students might initially predict that Bag Z has the best chance for picking a black marble since it contains the largest number of black marbles. However, by modeling the bags in *PE* (see figure 3), the students can then run simulations and make inferences based on the randomly generated data.

A key aspect of probabilistic situations is that outcomes based on a small number of trials do not often reflect what is expected based on theoretical probability. Consider the data shown in figure 4 that illustrates the distribution of black and white marbles after 20 trials and 200 trials for each bag. After 20 simulated with-replacement trials, 3 white and 17 black marbles were randomly selected from both Bag Y and Bag Z. This may lead some students to wonder if these bags have equal chances for picking a black marble. However, if the students repeatedly run sets of 20 trials from each bag, they should notice the variation and inconsistency in the results. This may lead students (or be suggested by the teacher) to run 200 trials and compare results. As shown in figure 4, the distribution after 200 trials suggests that Bag X and Bag Z may have similar chances for picking a black marble. Repeated trials of 200 or larger should help students see the stability and smaller variation in the distribution of colors. Since the pie graph can be shown changing during the simulation process, the decrease in variability can actually be seen graphically as the pie graph appears to “even out” and remain relatively stable after a large number of trials.

A simulation approach to the comparison of probabilities brings in a much richer perspective of probability and engages students in reasoning about the effect of sample size and the relationship between empirical and theoretical probabilities. In addition, the students can use the data to make inferences about the likelihood of picking a black marble from each bag and develop a data-based argument for why Bag X and Bag Z may have the same chance of picking a black marble. This argument can lead into a whole class discussion about looking for similarities and differences between each of the bags

that may lead students to think about the multiplicative structure between $\frac{3}{4}$ and $\frac{6}{8}$ and how to use proportional reasoning to justify that the two fractions are equivalent.

Sample Task 2: Create a Bag with Same Chance

After students have had an initial exploration such as sample task 1, they are ready to reason further about how to create a bag that has the same chance for picking a black marble as a given bag. Given a bag with 2 white and 4 black marbles, students can use *PE* to create and simulate an experiment with a bag of marbles that contains a different amount of black and white marbles but yet has the same chance of picking a black marble as the given bag. The students might use an approach similar to the one described in sample task 1. However, if left as an open-ended task, it is highly likely that students will find different ways to create a bag of marbles so that it has a $\frac{4}{6}$ chance of picking a black marble (e.g., 1 white & 2 black, 3 white & 6 black, 10 white & 20 black). This can lead nicely into a class discussion of an equivalent class of ratios and fractions and the infinite number of bags that can be created that maintain the multiplicative relationship between the fractions. Such an examination of equivalent fractions can even lead into a discussion of algorithms for generating a fraction equivalent to a known fraction.

Sample Task 3: Missing Black Marbles

The third marble bag task is a variation of the typical “missing value” proportional problem. To help students make a connection between objects in a bag and the expression of probabilities as ratios, fractions, and percents, *PE* has a Weight Tool¹

that will display the contents of a marble bag in each of these formats. Consider asking students to solve the following task:

Bag J has 4 black and 2 yellow marbles. If Bag K already contains 5 yellow marbles, how many black marbles must you add to Bag K so that it has the same chance of picking a black marble as Bag J?

As shown in figure 5, students can model this problem in PE and use the Weight Tool to display the changes in the probabilities for Bag K as black marbles are added to the bag. Once students think they have obtained the same proportion as Bag J, they can run simulations to confirm or disconfirm that the two bags lead to similar data distributions with a large number of trials. Again, a modeling and simulation approach to this “missing value” task can maintain the probabilistic nature of the task and develop into a class discussion of creating an algorithm for finding a missing value in a proportional situation.

Sample Task 4: Simulating Births

After focusing on developing students’ proportional reasoning through simulations of the marble tasks, it will be interesting to see if students use their sense of the effect of sample size or inappropriately apply their proportional reasoning to the Hospital Task in figure 1. Students should make an initial prediction and justification for whether the chance of having 4 girls out 5 births is more, less, or equal to the chance of 20 girls out of 25 births. This task is different from the marble tasks since the theoretical probability for a girl is assumed to be $\frac{1}{2}$ and the probability comparison is between empirical probabilities with different sample sizes rather than a comparison between the

theoretical and empirical probabilities. Thus, the issue of decrease in variability as sample size increases becomes a critical factor.

Once students make their initial predictions, they should use *PE* to repeatedly simulate both situations of 5 and 25 births and make observations and analysis of the data collected (see figure 6). Based on simulating sets of 5 and 25 births, students can infer that the distribution of 1 boy and 4 girls occurs on a frequently while 5 boys and 20 girls rarely occurs. Students may also conjecture that the result of 1 girl and 4 boy has the same chance as 1 boy and 4 girls. This conjecture can lead into more formal work with combinations to illustrate this equivalence. It is also useful to have a class discussion about the differences in variability in percents (as shown in the pie graph) of births that are girls between the two sample sizes of 5 and 25. Such simulation experiences and class discussions can help students appreciate the context of the probability task before applying proportional reasoning inappropriately.

Summary and Reflections

Proportional reasoning is an inherent part of developing a deep understanding of probability; however probability does not merely need to be an application of proportional reasoning. The study of probability through the use of simulations tools such as *PE* can build students' proportional reasoning abilities within the complexities of learning to reason probabilistically. Students can learn to use these types of reasoning together and build a robust understanding of equivalent ratios and fractions as well as the importance of sample size when making inferences based on data and comparing probabilities. The suggested activities with *PE* are aligned with the NCTM (2000)

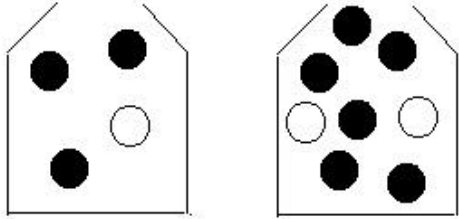
recommendation to use concepts of rational numbers and proportionality as a central unifying theme in middle school curricula. The study of probability no longer needs to be “saved” as an end-of-the-year unit where students “apply” rational number and proportion skills. Instead, probability can be used as a vehicle to develop a more meaningful and contextual understanding of rational number concepts and robust use of proportional reasoning.

¹Note: In more open-ended explorations, students can actually change the probability of an event happening directly in the weight tool. For example, students could model weather predictions and use the Weight Tool to enter the likelihood of rain and sunshine based on a weather forecast.

References

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Bag A Bag B

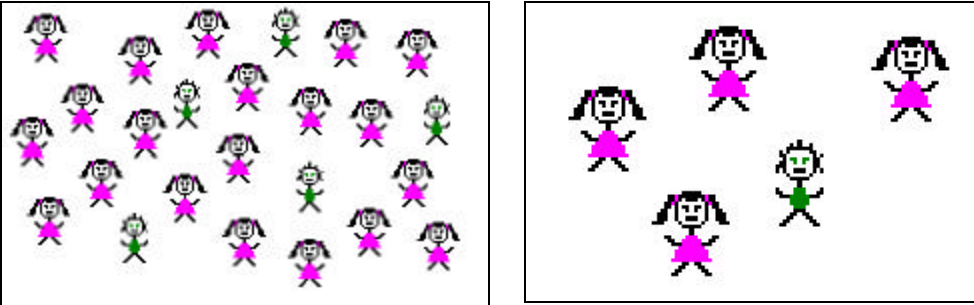
Which bag has the best chance for picking a black marble?

A) Bag A
B) Bag B
or
C) Bag A and Bag B have the same chance for picking a black marble

Explain Why

Figure 1. Marble Task to compare likelihood of an event in two different situations

Suppose that two hospitals kept track of the number of babies born each week. Hospital A is in a big city while Hospital B is in a small town.



20 Females and 5 Males
Hospital A

4 Females and 1 Male
Hospital B

Assuming that for any given birth it is equally likely for a boy or girl to be born, which do you think is more likely to occur?

a) 20 out of 25 of the babies born in Hospital A are female,
b) 4 out of 5 of the babies born in Hospital B are female,
or
c) Events a) and b) are equally likely to occur.

Figure 2. Hospital Task to compare likelihood of distributions of births. (adapted from Tversky & Kahneman, 1972)

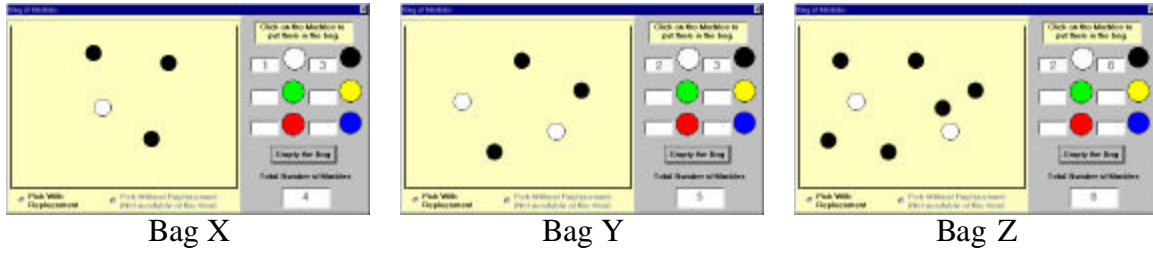


Figure 3. Bags of marbles used in Sample Task 1.

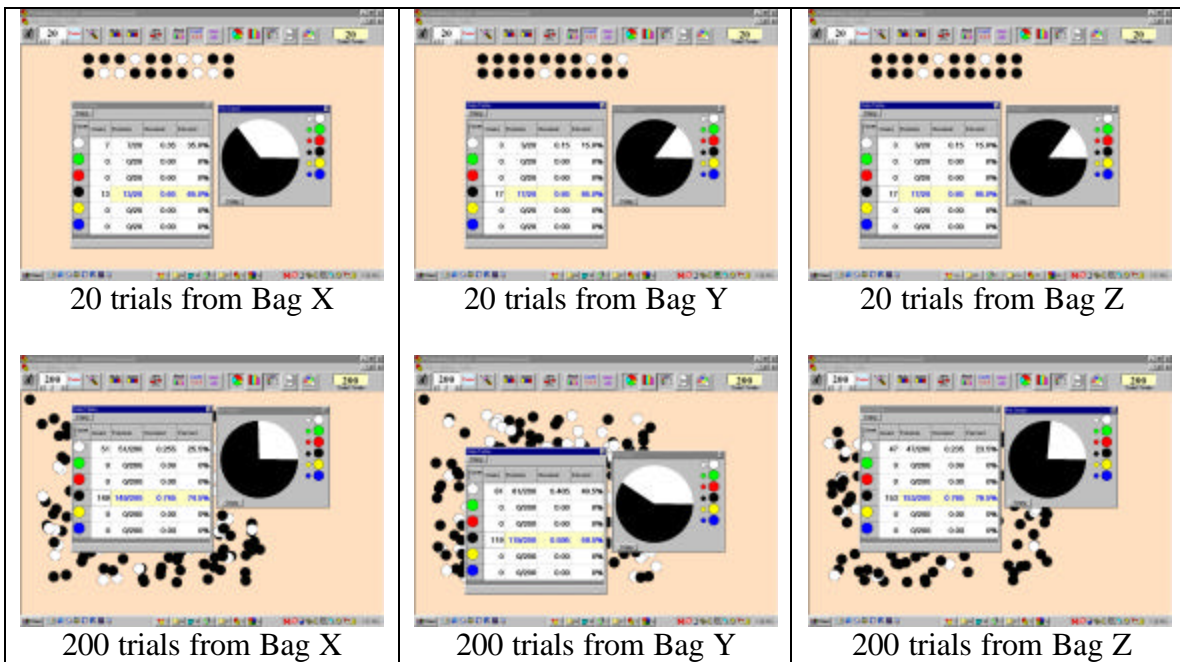


Figure 4. Sample Task 1 data collected with each bag for 20 and 200 trials.

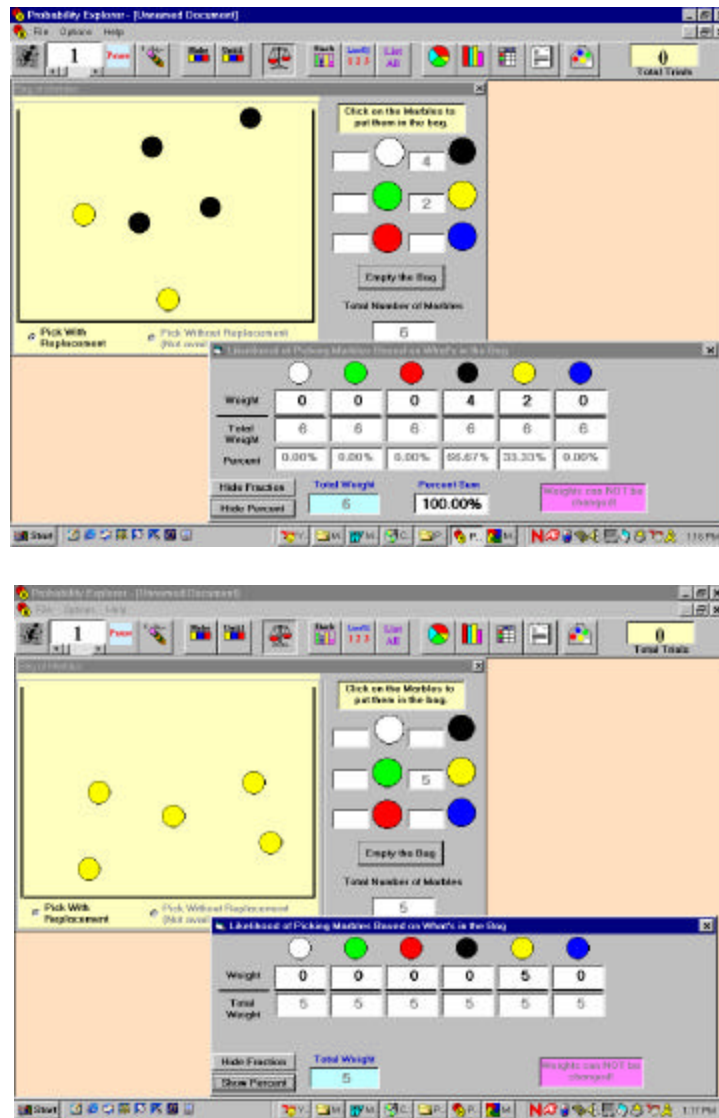


Figure 5. Using the Bag of Marbles and Weight Tool in *PE* to solve Sample Task 3

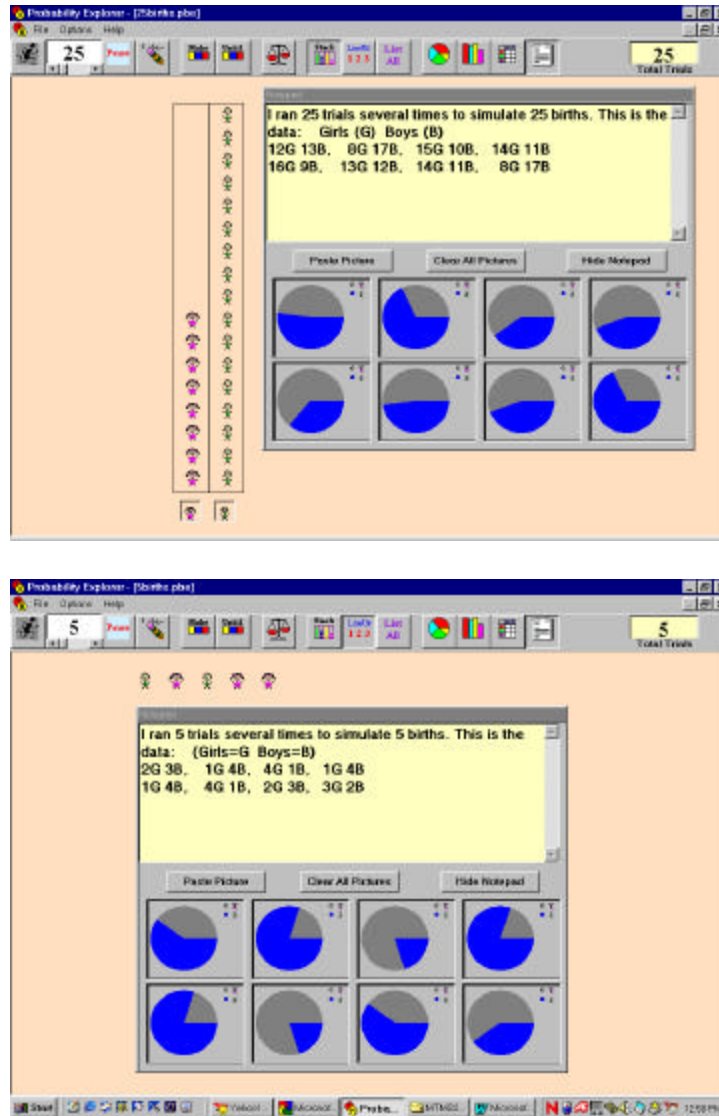


Figure 6. Data and pie graphs from simulations of 25 births and 5 births.